T1

若 $\sum_{n=1}^{\infty} u_n(x)$一致收敛，则设$\{S_n(x)\}$为其部分和函数列，$S(x)$为其和函数，

有$\forall\epsilon > 0$，$\exists N(\epsilon) > 0, s.t. \forall x \in I, n > N$时，有$|S_n(x) - S(x)| < \epsilon$

若 $\int_a^{+\infty} f(x,y)\mathrm{d}y$在$I$上一致收敛，则

$\forall\epsilon > 0, \exists N(\epsilon) > 0, s.t. \forall x \in I, n > N$时，有$\left|\int_n^{+\infty} f(x,y)\mathrm{d}y\right| < \epsilon$

T2

(1)设$f(x) = \sin x - x + \frac{1}{6}x^3 (x \geq 0)$

$$f'(x) = \cos x - 1 + \frac{1}{2}x^2, f''(x) = -\sin x + x \geq 0$$

则$f'(x) \geq f'(0) = 0, f(x) \geq f(0) = 0$

$\therefore f(\frac{1}{n}) \geq 0$，即 $\sin\frac{1}{n} \geq \frac{1}{n} - \frac{1}{6n^3}$

$\frac{1}{n^{2n\sin\frac{1}{n}}} \leq \frac{1}{n^{2-\frac{1}{3n^2}}} \leq \frac{1}{n^{\frac{5}{3}}}$，故由比较原则得该级数收敛

该级数为正项级数，则该级数绝对收敛

(2)设$r = \sqrt{x^2+y^2+z^2}$，有$\frac{\partial u}{\partial x} = \frac{x}{r}, \frac{\partial u}{\partial y} = \frac{y}{r}, \frac{\partial u}{\partial z} = \frac{z}{r}$

令$\vec{l} = (x'(t), y'(t), z'(t))\Big|_{(1,2,3)} = (1, 4t, 12t^3)\Big|_{(1,2,3)} = (1,4,12)$

则设$\overrightarrow{l_0} = \frac{\vec{l}}{|\vec{l}|} = \frac{(1,4,12)}{\sqrt{161}}, \nabla u = \frac{(1,2,3)}{\sqrt{14}}$

$$\frac{\partial u}{\partial \vec{l}} = \overrightarrow{l_0} \cdot (\nabla u) = \frac{45}{7\sqrt{46}}$$

T3

(1)法一：

$$V = \int_0^{\frac{1}{\sqrt{2}}} \mathrm{d}z \iint_{x^2+y^2 \leq z^2} \mathrm{d}x\mathrm{d}y + \int_{\frac{1}{\sqrt{2}}}^1 \mathrm{d}z \iint_{x^2+y^2 \leq 1-z^2} \mathrm{d}x\mathrm{d}y$$

$$= \int_0^{\frac{1}{\sqrt{2}}} \pi z^2 \mathrm{d}z + \int_{\frac{1}{\sqrt{2}}}^1 \pi(1-z^2)\mathrm{d}z$$

$$= \pi(\frac{1}{6\sqrt{2}} - 0 + (1 - \frac{1}{3}) - (\frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}})) = \frac{(2-\sqrt{2})}{3}\pi$$

法二：

$$V = \iint_{x^2+y^2 \leq \frac{1}{2}} \mathrm{d}x\mathrm{d}y \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} \mathrm{d}z$$

$$= \iint_{x^2+y^2 \leq \frac{1}{2}} (\sqrt{1-x^2-y^2} - \sqrt{x^2+y^2})\mathrm{d}x\mathrm{d}y$$

$$= \int_0^{2\pi} \mathrm{d}\theta \int_0^{\frac{1}{\sqrt{2}}} (\sqrt{1-r^2} - r)r\mathrm{d}r$$

$$= 2\pi(-\frac{1}{3}(\sqrt{1-r^2})^3 - \frac{r^3}{3})\Big|_0^{\frac{1}{\sqrt{2}}}$$

$$= 2\pi(-\frac{1}{6\sqrt{2}} + \frac{1}{3} - \frac{1}{6\sqrt{2}}) = \frac{(2-\sqrt{2})}{3}\pi$$

T3

(2)令 $x=y=\dfrac{\sin\theta}{\sqrt{2}}, z=\cos\theta(\theta\in[0,2\pi])$

$$\mathrm{d}s=\sqrt{2(\frac{\cos\theta}{\sqrt{2}})^2+(-\sin\theta)^2\mathrm{d}\theta}=\mathrm{d}\theta$$

$$\oint_L (x^2+y^2+z)^2 ds$$

$$=\int_0^{2\pi}(\sin^2\theta+\cos\theta)^2\mathrm{d}\theta$$

$$\xlongequal{\theta=\pi+\alpha}\int_{-\pi}^{\pi}(\sin^2\alpha-\cos\alpha)^2\mathrm{d}\alpha$$

$$=2\int_0^{\pi}(\sin^2\alpha-\cos\alpha)^2\mathrm{d}\alpha$$

$$\xlongequal{\alpha=\frac{\pi}{2}+\beta}2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos^2\beta+\sin\beta)^2\mathrm{d}\beta$$

$$=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin^4\beta-2\sin^3\beta-\sin^2\beta+2\sin\beta+1)\mathrm{d}\beta$$

$$=4\int_0^{\frac{\pi}{2}}(\sin^4\beta-\sin^2\beta+1)\mathrm{d}\beta$$

$$=4(\frac{3}{4}\cdot\frac{1}{2}-\frac{1}{2}+1)\cdot\frac{\pi}{2}=\frac{7}{4}\pi$$

T3

(3)$P=\dfrac{-y}{3x^2+4y^2}, Q=\dfrac{x}{3x^2+4y^2}$

在任意非原点处，有 $\dfrac{\partial P}{\partial y}=\dfrac{4y^2-3x^2}{(3x^2+4y^2)^2}=\dfrac{\partial Q}{\partial x}$

取一个很小的椭圆曲线 $L':3x^2+4y^2=\delta^2,\delta<0.001$

则 L' 完全在 L 内，设 L' 和 L 间的区域为 S, L' 包围的椭圆为 S'

$$\int_L\frac{x\mathrm{d}y-y\mathrm{d}x}{3x^2+4y^2}$$

$$=\iint_S(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})\mathrm{d}x\mathrm{d}y+\int_{L'}\frac{x\mathrm{d}y-y\mathrm{d}x}{3x^2+4y^2}$$

$$=\frac{1}{\delta^2}\int_{L'}x\mathrm{d}y-y\mathrm{d}x$$

$$=\frac{1}{\delta^2}\iint_{S'}(1+1)\mathrm{d}x\mathrm{d}y$$

$$=\frac{1}{\delta^2}\pi\frac{\delta}{\sqrt{3}}\frac{\delta}{2}=\frac{\pi}{2\sqrt{3}}$$

T3

(4)令 $x=\sin\varphi\cos\theta, y=2\sin\varphi\sin\theta, z=3\cos\varphi$

$\varphi\in[0,\dfrac{\pi}{2}],\theta\in[0,2\pi]$

$$\frac{\partial(x,y)}{\partial(\varphi,\theta)}=2\sin\varphi\cos\varphi=\sin2\varphi\geq0$$

$$\frac{\partial(y,z)}{\partial(\varphi,\theta)}=6\sin^2\varphi\cos\theta$$

$$\iint_S x^3\mathrm{d}y\mathrm{d}z$$

$$=\iint_{D_{\varphi\theta}}\sin^3\varphi\cos^3\theta\cdot6\sin^2\varphi\cos\theta\mathrm{d}\varphi\mathrm{d}\theta$$

$$=6\int_0^{\frac{\pi}{2}}\sin^5\varphi\mathrm{d}\varphi\int_0^{2\pi}\cos^4\theta\mathrm{d}\theta$$

$$=24\cdot\frac{3}{4}\frac{1}{2}\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\sin^5\varphi\mathrm{d}\varphi$$

$$=\frac{9\pi}{2}\cdot\frac{4}{5}\frac{2}{3}=\frac{12}{5}\pi$$

T4

(1) $\begin{cases} x-y=1-z \\ x+y=1-2z \end{cases} \Rightarrow \begin{cases} x=1-\frac{3}{2}z \\ y=-\frac{z}{2} \end{cases}$

令 $g(z)=f(x(z),y(z),z)$，得 $g(z)=\dfrac{3(|z|+|z-\frac{2}{3}|)}{2} \geq \dfrac{3}{2}|z-(z-\dfrac{2}{3})|=1$

当 $z\in[0,\dfrac{2}{3}]$时取等，则所有极小值点为：

$(1-\dfrac{3}{2}z,-\dfrac{z}{2},z),z\in[0,\dfrac{2}{3}]$

(2)即令 $z=0$，有 $|y|=|z|=0$，仅 $|x|$为非零项

所求极小值点即为 $(1,0,0)$

T5

法一 :设 L为区域 $D=[0,1]\times[0,1]$的边界曲线

有 $0=\displaystyle\int_L xyf'_y(x,y)\mathrm{d}y-xyf'_x(x,y)\mathrm{d}x=\iint_D(yf'_y+xf'_x+2xyf''_{xy})\mathrm{d}x\mathrm{d}y$

$$\therefore \iint_D xyf''_{xy}(x,y)\mathrm{d}x\mathrm{d}y=\frac{1}{2}\iint_D[yf'_y(x,y)+xf'_x(x,y)]\mathrm{d}x\mathrm{d}y$$

$$0=\int_L xf(x,y)\mathrm{d}y-yf(x,y)\mathrm{d}x=\iint_D(xf'_x+yf'_y+2f)\mathrm{d}x\mathrm{d}y$$

$$\therefore \iint_D f(x,y)\mathrm{d}x\mathrm{d}y$$
$$=\frac{1}{2}\iint_D[yf'_y(x,y)+xf'_x(x,y)]\mathrm{d}x\mathrm{d}y=\iint_D xyf''_{xy}(x,y)\mathrm{d}x\mathrm{d}y$$

故得证

法二 :利用累次积分交换积分顺序和分部积分进行变换

$$\iint_D xyf_{xy}(x,y)\mathrm{d}x\mathrm{d}y$$
$$=\int_0^1 x\mathrm{d}x\int_0^1 yf_{xy}(x,y)\mathrm{d}y$$
$$=\int_0^1 x\mathrm{d}x\int_0^1 y\mathrm{d}f_x(x,y)$$
$$=\int_0^1 x\mathrm{d}x\left(-\int_0^1 f_x(x,y)\mathrm{d}y\right)\text{(分部积分)}$$
$$=-\int_0^1 \mathrm{d}x\int_0^1 xf_x(x,y)\mathrm{d}y$$
$$=-\int_0^1 \mathrm{d}y\int_0^1 xf_x(x,y)\mathrm{d}x$$
$$=-\int_0^1 \mathrm{d}y\left(-\int_0^1 f(x,y)\mathrm{d}x\right)\text{(分部积分)}$$
$$=\iint_D f(x,y)\mathrm{d}x\mathrm{d}y$$

T6

(1)$|a_n\cos nx+b_n\sin nx|\le|a_n||\cos nx|+|b_n||\sin nx|\le|a_n|+|b_n|\le\dfrac{M}{n^3}$

由比较原则得该三角级数收敛

(2)由 (1)，该三角级数收敛，则设 $f(x)=\dfrac{a_0}{2}+\sum\limits_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$

$\Big|a_n\cos nx+b_n\sin nx\Big|\le|a_n|+|b_n|\le\dfrac{M}{n^3}$，而正项级数 $\sum\dfrac{M}{n^3}$收敛

由$Weierstrass$判别法，该三角级数一致收敛，故对该级数积分可逐项求积，按定义验证如下：

$$\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=a_0+\frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\mathrm{d}x$$

$$=a_0+\frac{1}{\pi}\sum_{n=1}^{\infty}(a_n\int_{-\pi}^{\pi}\cos nx\mathrm{d}x+b_n\int_{-\pi}^{\pi}\sin nx\mathrm{d}x)=a_0$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\mathrm{d}x=\frac{1}{\pi}\sum_{k=1}^{\infty}a_k\int_{-\pi}^{\pi}\cos kx\cos nx\mathrm{d}x+b_k\int_{-\pi}^{\pi}\sin kx\cos nx\mathrm{d}x=a_n$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\mathrm{d}x=\frac{1}{\pi}\sum_{k=1}^{\infty}a_k\int_{-\pi}^{\pi}\cos kx\sin nx\mathrm{d}x+b_k\int_{-\pi}^{\pi}\sin kx\sin nx\mathrm{d}x=b_n$$

$$\left(\int_{-\pi}^{\pi}\cos kx\cos nx\mathrm{d}x=\int_{-\pi}^{\pi}\frac{\cos(k-n)x+\cos(k+n)x}{2}\mathrm{d}x=\frac{1}{2}\int_{-\pi}^{\pi}\cos(k-n)x=\begin{cases}\pi,k=n\\0,k\ne n\end{cases}\right)$$

$$\left(\int_{-\pi}^{\pi}\sin kx\sin nx\mathrm{d}x=\int_{-\pi}^{\pi}\frac{\cos(k-n)x-\cos(k+n)x}{2}\mathrm{d}x=\frac{1}{2}\int_{-\pi}^{\pi}\cos(k-n)x=\begin{cases}\pi,k=n\\0,k\ne n\end{cases}\right)$$

$$\left(\int_{-\pi}^{\pi}\sin px\cos qx\mathrm{d}x=\int_{-\pi}^{\pi}\frac{\sin(p+q)x-\sin(p-q)x}{2}\mathrm{d}x=0\right)$$

故证得该为$Fourier$级数

T6

(3)(2)中已经证得该三角级数一致收敛，则对其求导可逐项求导，有

$$f'(x)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\right)$$

$$=\sum_{n=1}^{\infty}\left(\frac{\mathrm{d}}{\mathrm{d}x}(a_n\cos nx+b_n\sin nx)\right)=\sum_{n=1}^{\infty}n(b_n\cos nx-a_n\sin nx)$$

$\Big|n(b_n\cos nx-a_n\sin nx)\Big|\le n(|a_n|+|b_n|)\le\dfrac{M}{n^2}$，正项级数 $\sum\dfrac{M}{n^2}$收敛

由$Weierstrass$判别法，级数 $\sum\limits_{n=1}^{\infty}n(b_n\cos nx-a_n\sin nx)$一致收敛

而级数每一项 $n(b_n\cos nx-a_n\sin nx)$都连续，故其和函数 $f'(x)$连续，得证

T7

缺初始条件 $F(x_0,y_0)=0$。

如果补上初始条件，应该就是默写隐函数存在唯一性定理

T8

(1)令 $f(x)=\cos(\dfrac{\pi}{2}x^{\frac{1}{n}})+\dfrac{\pi}{n}(x-1),x\in[\dfrac{1}{2},1]$

有 $f'(x)=\dfrac{\pi}{2n}(2-x^{\frac{1}{n}-1}\sin(\dfrac{\pi}{2}x^{\frac{1}{n}}))\ge\dfrac{\pi}{2n}(2-(\dfrac{1}{2})^{-1})=0$

故 $f(x)\le f(1)=0\Rightarrow\cos(\dfrac{\pi}{2}x^{\frac{1}{n}})\le\dfrac{\pi}{n}(1-x),x\in[\dfrac{1}{2},1]$

(2)$x\in[0,1]$,有 $\left|\cos(\dfrac{\pi}{2}x^{\frac{1}{n}})\dfrac{x^n}{n^2}\right|\le\dfrac{1}{n^2}$，正项级数 $\sum\dfrac{1}{n^2}$收敛

(优级数判别法)

故 $\sum\limits_{n=1}^{\infty}\cos(\dfrac{\pi}{2}x^{\frac{1}{n}})\dfrac{x^n}{n^2}$在 $[0,1]$一致收敛