Introduction to Theoretical Computer Science

2025-2026 秋冬期末

图灵回忆卷

1) Write T if the statement is true, and F if it is false.
   a) [] If A is computable, then A^* is computable.
   b) [] $\exists$ a boolean function $F : \{0,1\}^m \to \{0,1\}^n$, such that there exists a circuit with at least $\Omega(m2^n)$ gates that computes F.
   c) [] Gate set {OR, NOT} is universal, in the sense that any boolean function can be computed by a circuit using only these 2 kinds of gates.
   d) [] To simulate multiple TMs with k states in total using one TM, the resulting TM needs $\Omega(k)$ states.
   e) [] To convert an NFA with k states to a DFA, the resulting DFA has $\Theta\left(2^k\right)$ states.
   f) [] Let a DFA with k states recognize a regular language L. L contains an infinite number of strings if and only if L contains a string of length at least k.
   g) [] There exists a randomized algorithm that computes some problem with error rate at most $\frac{1}{3}$ if and only if there exists a randomized algorithm that computes the same problem with error rate at most $\frac{1}{2026}$.
   h) [] Probablistic Turing Machine is more powerful than normal Turing Machine, in the sense that it can compute some functions that normal Turing Machine cannot compute.
   i) [] If $f, g \in \mathcal{P}$ and they are not constant functions, then $f \underset{p}{\leq} g$.
   j) [] If $f \underset{p}{\leq} g$ and $g \in \mathcal{BPP}$, then $f \in \mathcal{BPP}$.

2) (Regular Language and Automaton) Judge whether the following languages are regular. Prove your answer.
   a) $L_1 = \{wx \in \{0,1\}^* \mid x \text{ as a binary number}, x \bmod 3 = 2\}$.
   b) $L_2 = \{w0x \in \{0,1\}^* \mid x \text{ has strictly more 1s than } w\}$.

3) (Context-Free Language and Pushdown Automaton) Show that the language $L = \left\{w101x \in \{0,1\}^* \mid w^R \text{ is a subsequence of } x\right\}$ is a context-free language. (Note: **Subsequence** is not the same as **Substring**. For example, "abc" is a subsequence of "axbxc", but not a substring of it.)

4) (Automaton) Let A be a context-free language and B be a regular language. Prove that $A \setminus B = \{w \in \{0,1\}^* \mid w \in A \wedge w \notin B\}$ is a context-free language. (Hint: You may conceptually run a PDA and a DFA in parallel.)

5) (Reduction and Uncomputability) $f : \{0,1\}^* \to \{0,1\}$ is a function which takes 2 Turing Machines M_1 and M_2 as input, and $f(M_1, M_2) = 1$ if there exist at least 2026 strings on which M_1 and M_2 both halt. Prove that f is uncomputable by reducing HALT to it. (Hint: You may let one of M_1 and M_2 halt on every input.)

6) （存疑）(Time Complexity Classes) WSAT problem, based on SAT problem, is defined as follows: Given a boolean formula φ in CNF form and every instance is OR of one to three literals (**Note: In WSAT literals are only allowed to be variables themselves and their negations are not allowed**), and an integer k, decide whether there exists a satisfying assignment to φ such that **exactly** k variables are assigned to 1.
   a) Prove that WSAT $\in \mathcal{NP}$.
   b) Prove that VC $\underset{p}{\leq}$ WSAT, where VC is the Vertex Cover problem.